

Dissipative and nonequilibrium effects near a superconductor-metal quantum critical point

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We present a microscopic derivation of the effect of current flow on a system near a superconductor-metal quantum critical point. The model studied is a 2d itinerant electron system where the electrons interact via an attractive interaction and are coupled to an underlying normal metal substrate which provides a source of dissipation, and also provides a source of inelastic scattering that allows a nonequilibrium steady state to reach. A nonequilibrium Keldysh action for the superconducting fluctuations on the normal side is derived. Current flow, besides its minimal coupling to the order parameter is found to give rise to two new effects. One is a source of noise that acts as an effective temperature $T_{eff} = eEv_F\tau_{sc}$ where E is the external electric field, v_F the Fermi velocity, and τ_{sc} is the escape time into the normal metal substrate. Secondly current flow also produces a drift of the order-parameter. Scaling equations for the superconducting gap and the current are derived and are found to be consistent with previous phenomenological treatments as long as a temperature $T \sim T_{eff}$ is included. The current induced drift is found to produce additional corrections to the scaling which are smaller by a factor of $\mathcal{O}(\frac{1}{E_F\tau_{sc}})$, E_F being the Fermi energy.

I. INTRODUCTION

Quantum critical phenomena is the study of how a system loses long range order at $T \rightarrow 0$ as a parameter of the Hamiltonian is changed^{1,2}. The non-commutativity of position and momentum in quantum mechanics implies that the spatial and temporal fluctuations of the order parameter are coupled to each other at the zero temperature quantum critical point. The effect of temperature on a quantum critical point has some generic features^{3,4} such as, a non-zero temperature produces dephasing or decoherence that cuts off divergences in correlation lengths and times. Thermal decoherence also decouples spatial and temporal fluctuations causing a crossover from quantum to classical behavior.

While quantum phase transitions for systems in equilibrium have been extensively studied, a much less understood issue is the effect of a nonequilibrium probe such as current flow on a system in the vicinity of a quantum critical point. Scaling theories exist which assume that the primary effect of a nonequilibrium probe is to produce decoherence or an effective temperature. Thus nonequilibrium scaling relations are obtained by replacing temperature in the equilibrium scaling relations by the appropriate nonequilibrium energy scale¹. A microscopic treatment to justify this and in the process also identify the appropriate nonequilibrium energy scale is often challenging as this requires a treatment that goes beyond a linear response Kubo formula calculation. Only a handful of such treatments exist for magnetic-paramagnetic^{5,6,7,8}, and superfluid-insulator/metal quantum critical points^{9,10,11}.

In this paper we revisit the problem of non-linear effects, in particular the effect of a uniform current flow on a system near a superconductor-metal quantum critical point. Existing studies have so far involved writing phenomenological effective theories for a charged order-parameter in the presence of an electric field and/or external dissipation^{9,10}. In this paper we carry out a fully microscopic derivation of the appropriate nonequilibrium effective theory starting from a fermionic model under external drive. In doing so, we address the issue of how the fermionic system reaches a nonequilibrium steady state, and find that the underlying nonequilibrium fermions give rise to additional terms in the effective theory for the charged order-parameter that were previously missed. We then proceed to determine the effect of these terms on scaling near the equilibrium quantum critical point. (Note that the observation that nonequilibrium electrons can significantly modify the scaling near critical points was also pointed out in¹, and was experimentally observed in thin films of Bi in¹²).

The geometry that will be studied (shown schematically in Fig 1) is a 2d itinerant electron system where the electrons interact with each other via an attractive interaction. This system is driven out of equilibrium by an in-plane electric field so that a current flows through the bulk of the system. In addition, the 2d layer is coupled to an underlying normal metal substrate with which it can exchange particle as well as energy which thus serves as a heat sink that allows the layer to reach a nonequilibrium steady state. The coupling to the substrate also provides a source of dissipation that when made sufficiently large can destroy superconductivity in the layer¹¹. (Note that the model in Fig 1 for the case of repulsive interactions of the electrons in the layer and for parameters that are such that the system is near a ferromagnetic-paramagnetic quantum critical point was studied in⁸).

As we shall show, in equilibrium the effective action for the superconducting fluctuations on the ordered and the disordered side has a local Caldeira-Leggett dissipation typical of systems where particle number is not conserved¹³.

Effective theories for superconducting fluctuations with local dissipation have been extensively studied in equilibrium,^{14,15,16,17} but hardly at all out of equilibrium (with the exception of⁹). Out of equilibrium, our microscopic treatment reveals three effects of current flow, one is the usual minimal coupling of the current to the charged order-parameter, second is a source of noise which for low frequencies and long wavelength fluctuations of the order-parameter essentially acts as an effective temperature which equals the typical energy an electron gains on being accelerated by the external electric field. In our model $T_{eff} = eEv_F\tau_{sc}$, where e and v_F are the charge and Fermi velocity of the electrons, and τ_{sc} is the inelastic scattering time or escape time into the normal metal reservoirs. Thirdly, we also find that current flow can cause the order-parameter to drift with a drift velocity $v_D = \frac{eE}{m}\tau_{sc}$, m being the mass of the electrons.

We briefly mention the relation of the work presented here to that of⁹ which also had an extrinsic dissipation which was introduced phenomenologically. Thus in their model, current affects the order-parameter only via its minimal coupling to it, and the properties of the dissipative reservoir were unaffected by the current flow. In our model, the dissipation originates via the coupling of the superconducting order-parameter to the underlying normal electrons, whose properties are itself modified due to an external drive. Taking this effect into account shows that the order-parameter is subjected to a noise and also drifts with the current. As we shall show, in the quantum-critical regime, current noise gives corrections to the scaling which are of $\mathcal{O}((T_{eff}\tau_{sc})^{2/3})$, where $T_{eff}\tau_{sc}$ is the ratio of the typical energy gained from the electric field between collisions, and the energy lost due to inelastic scattering. Since, in our model τ_{sc} is largely independent of the electric field (it may acquire some corrections at large electric fields), the corrections to scaling due to noise $((T_{eff}\tau_{sc})^{2/3} \ll 1)$ is subdominant in the quantum critical regime, with the dominant scaling behavior being that derived in⁹. In the quantum disordered regime however, direct coupling and noise effects are found to be equally important. Current drift on the other hand gives a correction which is smaller by an additional factor of $\mathcal{O}(1/E_F\tau_{sc})$ where E_F is the fermi energy.

The paper is organized as follows. The model is presented in Section II and is treated within a Keldysh path integral approach which will allow us to study out-of-equilibrium effects. We first study the equilibrium properties of the system by performing a mean-field treatment in section III which reveals a dissipation induced quantum critical point, which can also be understood as a proximity effect. A derivation of the effective action for the superconducting fluctuations about the equilibrium ordered side is presented in Appendix A, and the origin of a local Caldeira-Leggett dissipation arising due to nonconserved particle number is highlighted. Fluctuation about the nonequilibrium disordered state is studied in section V and the new terms in the bosonic theory corresponding to current noise and drift are derived. Scaling equations for the gap and the current are derived in section VI. Many of the details of the derivation have been relegated to the appendices. Finally we conclude in section VII where we discuss our results in the context of existing experiments.

II. MODEL

We consider a model of electrons in a 2d layer that interact via a short ranged attractive interaction responsible for a superconducting instability and are coupled via tunneling to a reservoir of non-interacting electrons. The Hamiltonian for the system is

$$H = H_{bath} + H_{layer} + H_{layer-bath} \quad (1)$$

where H_{layer} is the interacting electron layer whose critical properties we are interested in, H_{bath} represents the reservoir, while $H_{layer-bath}$ represents the coupling between the two.

$$H_{layer} = \sum_{\sigma} \psi_{\sigma}^{\dagger} \frac{1}{2m} \left(\frac{\vec{\nabla}}{i} - \frac{e}{\hbar c} \vec{A} \right)^2 \psi_{\sigma}^{\dagger} - \lambda \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} \psi_{\downarrow} \psi_{\uparrow} \quad (2)$$

$$H_{bath} = \sum_{k_z, k, \sigma} \epsilon_{k_z, k, \sigma}^b c_{k_z, k, \sigma}^{\dagger} c_{k_z, k, \sigma} \quad (3)$$

$$H_{layer-bath} = \sum_{\sigma, k_z, k} \left(t c_{k_z, k, \sigma}^{\dagger} \psi_{k\sigma} + h.c. \right) \quad (4)$$

σ is the spin label, c represent the reservoir electrons, k_z is the momentum transverse to the superconductor-bath interface and is not conserved on tunneling, while k is the momentum within the layer. We assume the superconductor-bath interface to be smooth, so that the in-plane momentum is conserved on tunneling. The schematic of the model is shown in Fig 1. In addition the electrons in the interacting layer are subjected to a dc electric field which we represent via a vector potential $A = -cEt$. (We will set $\hbar = 1$).

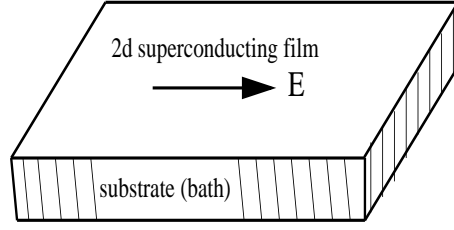


FIG. 1: A 2d itinerant electron system near a superconducting instability and driven out of equilibrium by application of an in-plane electric field. A steady state is reached via coupling to a normal metal substrate.

We write the Keldysh action for this model^{18,19,20},

$$Z_K = \int \mathcal{D} [\psi_{\pm\sigma}, \bar{\psi}_{\pm\sigma}, c_{\pm,\sigma}, \bar{c}_{\pm,\sigma}] e^{i \int_{-\infty}^{\infty} dt d^d x ([L_-^e + L_-^{res}] - [L_+^e + L_+^{res}])} \quad (5)$$

where \pm labels the Keldysh time-ordering, L^{res} is the action for the reservoir electrons, while L^e is the Keldysh action for the layer electrons and the coupling with the reservoir.

$$L_{\pm}^e = \sum_{\sigma} \bar{\psi}_{\pm\sigma} \left[i \frac{\partial}{\partial t} - \frac{1}{2m} \left(\frac{\vec{\nabla}}{i} - \frac{e}{c} \vec{A} \right)^2 + \mu \right] \psi_{\pm\sigma} + \lambda \bar{\psi}_{\pm\uparrow} \bar{\psi}_{\pm\downarrow} \psi_{\pm\downarrow} \psi_{\pm\uparrow} - \sum_{\sigma} t [\psi_{\pm\sigma}^{\dagger}(x) c_{\pm\sigma}(x) + h.c.] \quad (6)$$

$$L_{\pm}^{res} = \sum_{\sigma} \bar{c}_{\pm\sigma} \left[i \frac{\partial}{\partial t} - H_0 + \mu \right] c_{\pm\sigma} \quad (7)$$

where μ is a chemical potential.

We perform a Hubbard Stratonovich decoupling of the attractive interaction

$$\exp \left(i \lambda \int dt d^d x [\bar{\Psi}_{-\uparrow} \bar{\Psi}_{-\downarrow} \Psi_{-\downarrow} \Psi_{-\uparrow} - \bar{\Psi}_{+\uparrow} \bar{\Psi}_{+\downarrow} \Psi_{+\downarrow} \Psi_{+\uparrow}] \right) = \int \mathcal{D} [\Delta_{\pm}, \Delta_{\pm}^*] \exp \left(-i \int dt d^d x \frac{|\Delta_{-}|^2 - |\Delta_{+}|^2}{\lambda} \right) \exp \left(i \int dt d^d x (\Delta_{-} \bar{\Psi}_{-\uparrow} \bar{\Psi}_{-\downarrow} + \Delta_{-}^* \Psi_{-\downarrow} \Psi_{-\uparrow} - \Delta_{+} \bar{\Psi}_{+\uparrow} \bar{\Psi}_{+\downarrow} - \Delta_{+}^* \Psi_{+\downarrow} \Psi_{+\uparrow}) \right) \quad (8)$$

and in the process introduce the bosonic fields Δ_{\pm} which represent superconducting fluctuations. Using Nambu notation $\Psi_{\pm} = \begin{pmatrix} \psi_{\pm\uparrow} \\ \psi_{\pm\downarrow} \end{pmatrix}$, $\bar{\Psi}_{\pm} = (\bar{\psi}_{\pm\uparrow} \quad \bar{\psi}_{\pm\downarrow})$, $\hat{c}_{\pm} = \begin{pmatrix} c_{\pm\uparrow} \\ c_{\pm\downarrow} \end{pmatrix}$, $\hat{\bar{c}}_{\pm} = (\bar{c}_{\pm\uparrow} \quad \bar{c}_{\pm\downarrow})$ the Lagrangian becomes,

$$L_{\pm}^e = \bar{\Psi}_{\pm} \begin{pmatrix} i \frac{\partial}{\partial t} - \frac{1}{2m} \left(\frac{\vec{\nabla}}{i} - \frac{e}{c} \vec{A} \right)^2 + \mu & \Delta_{\pm} \\ \Delta_{\pm}^* & i \frac{\partial}{\partial t} + \frac{1}{2m} \left(\frac{\vec{\nabla}}{i} + \frac{e}{c} \vec{A} \right)^2 - \mu \end{pmatrix} \Psi_{\pm} - t \left[\bar{\Psi}_{\pm} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{c}_{\pm} + h.c. \right] - \frac{|\Delta_{\pm}|^2}{\lambda} \quad (9)$$

The electronic degrees of freedom may now be formally integrated out, resulting in a Keldysh action entirely in terms of the fluctuating fields Δ_{\pm} . A rotation to retarded, advanced, Keldysh space leads to¹⁹

$$Z_K = \int \mathcal{D} [\Delta_{q,cl}, \Delta_{q,cl}^*] \exp (Tr \ln \mathcal{G}^{-1}) \exp \left(-i \int dt d^d x \left[\frac{2\Delta_q^* \Delta_{cl} + 2\Delta_{cl}^* \Delta_q}{\lambda} \right] \right) \quad (10)$$

where $\Delta_q = \frac{\Delta_- - \Delta_+}{2}$, $\Delta_{cl} = \frac{\Delta_- + \Delta_+}{2}$ are respectively the quantum and classical components of the fluctuating fields. \mathcal{G} is a 4×4 matrix in Nambu and Keldysh $(\tau_{x,y,z})$ space which obeys the Dyson equation

$$\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + \begin{pmatrix} 0 & \Delta_{cl} \\ \Delta_{cl}^* & 0 \end{pmatrix} \otimes \tau_0 + \begin{pmatrix} 0 & \Delta_q \\ \Delta_q^* & 0 \end{pmatrix} \otimes \tau_x \quad (11)$$

where \mathcal{G}_0 is the exact Green's function for non-interacting electrons coupled to reservoirs and subjected to an external electric field. The full Green's function \mathcal{G} may be written as follows in Nambu and Keldysh space, $\mathcal{G} = \begin{pmatrix} \mathcal{G}^R & \mathcal{G}^K \\ 0 & \mathcal{G}^A \end{pmatrix}$

where the $\mathcal{G}^{R,A,K}$ are the following 2×2 matrices,

$$\frac{1}{2}\mathcal{G}^R(t, t') = -i\langle\Psi_{cl}(t)\bar{\Psi}_q(t')\rangle = \frac{1}{2}\begin{pmatrix} G^R & F^R \\ \bar{F}^R & \bar{G}^R \end{pmatrix}$$

$$\frac{1}{2}\mathcal{G}^K(t, t') = -i\langle\Psi_{cl}(t)\bar{\Psi}_{cl}(t')\rangle = \frac{1}{2}\begin{pmatrix} G^K & F^K \\ \bar{F}^K & \bar{G}^K \end{pmatrix}$$

and the retarded Green's functions are defined as,

$$G^R(x, t; x', t') = -i\theta(t - t')\langle\{\psi_\uparrow(x, t), \bar{\psi}_\uparrow(x', t')\}\rangle = G_\uparrow^R(x, t; x', t') \quad (12)$$

$$\bar{G}^R(x, t; x', t') = -i\theta(t - t')\langle\{\bar{\psi}_\downarrow(x, t), \psi_\downarrow(x', t')\}\rangle = -G_\downarrow^A(x' t'; x, t) \quad (13)$$

$$F^R(x, t; x', t') = -i\theta(t - t')\langle\{\psi_\uparrow(x, t), \psi_\downarrow(x', t')\}\rangle \quad (14)$$

$$\bar{F}^R(x, t; x', t') = -i\theta(t - t')\langle\{\bar{\psi}_\downarrow(x, t), \bar{\psi}_\uparrow(x', t')\}\rangle \quad (15)$$

and the Keldysh Green's functions are

$$G^K(x, t; x', t') = -i\langle[\psi_\uparrow(x, t), \bar{\psi}_\uparrow(x', t')]\rangle = G_\uparrow^K(x, t; x', t') \quad (16)$$

$$\bar{G}^K(x, t; x', t') = -i\langle[\bar{\psi}_\downarrow(x, t), \psi_\downarrow(x', t')]\rangle = -G_\downarrow^K(x' t'; x, t) \quad (17)$$

$$F^K(x, t; x', t') = -i\langle[\psi_\uparrow(x, t), \psi_\downarrow(x', t')]\rangle \quad (18)$$

$$\bar{F}^K(x, t; x', t') = -i\langle[\bar{\psi}_\downarrow(x, t), \bar{\psi}_\uparrow(x', t')]\rangle \quad (19)$$

III. MEAN FIELD TREATMENT IN EQUILIBRIUM ($E = 0$)

The mean-field equations may be obtained by minimizing Eq. 10 with respect to the quantum (Δ_q) and classical (Δ_{cl}) fluctuations of the order-parameter. A Ginzburg-Landau action is then obtained by expanding the Keldysh functional in fluctuations about the mean field solution. We first outline these steps for the equilibrium case *i.e.*, $A = Et = 0$, before turning to the nonequilibrium case.

In equilibrium the single particle Green's function \mathcal{G}_0 may be easily obtained. In Fourier space the retarded Green's functions are,

$$G_{0R}^{-1} = \omega - \xi_k - \Sigma^R \quad (20)$$

$$\bar{G}_{0R}^{-1} = \omega + \xi_k - \bar{\Sigma}^R \quad (21)$$

where $\xi_k = \epsilon_k - \mu$, and the self-energies Σ arise due to coupling to reservoirs and have the form,

$$\Sigma^R = \sum_{k_z} \frac{t^2}{\omega - \epsilon_{k_z}^b - \epsilon_k^b + \mu + i\delta} \simeq -i\Gamma \quad (22)$$

$$\bar{\Sigma}^R = \sum_{k_z} \frac{t^2}{\omega + \epsilon_{k_z}^b + \epsilon_k^b - \mu + i\delta} \simeq -i\Gamma \quad (23)$$

where $\Gamma = \pi\rho t^2$, ρ being the density of states of the reservoirs. We have taken the reservoir dispersion in Eq. 3 to be $\epsilon_{k_z, k}^b = \epsilon_{k_z}^b + \epsilon_k^b$. Note that we will interchangeably use the notation

$$\tau_{sc} = \frac{1}{2\Gamma} \quad (24)$$

to represent the typical escape time into the reservoirs.

For a reservoir in equilibrium at temperature T , the Keldysh self energies of the layer electrons due to coupling to the reservoir obey the fluctuation-dissipation theorem,

$$\Sigma^K = -2i\Gamma \tanh \frac{\omega}{2T} \quad (25)$$

$$\bar{\Sigma}^K = -2i\Gamma \tanh \frac{\omega}{2T} \quad (26)$$

Moreover,

$$G_0^K = G_0^R \Sigma^K G_0^A \quad (27)$$

$$\bar{G}_0^K = \bar{G}_0^R \bar{\Sigma}^K \bar{G}_0^A \quad (28)$$

It therefore follows that

$$\mathcal{G}_{R0}^{-1} = \begin{pmatrix} \omega - \xi_k + i\Gamma & 0 \\ 0 & \omega + \xi_k + i\Gamma \end{pmatrix} \Rightarrow \mathcal{G}_{R0} = \frac{1}{(\omega + i\Gamma)^2 - \xi_k^2} \begin{pmatrix} \omega + \xi_k + i\Gamma & 0 \\ 0 & \omega - \xi_k + i\Gamma \end{pmatrix} \quad (29)$$

and the fluctuation-dissipation theorem is obeyed so that,

$$\mathcal{G}_{0K} = -2i\Gamma \left[\tanh \frac{\omega}{2T} \right] \mathcal{G}_{0R} \mathcal{G}_{0A} = (\mathcal{G}_{0R} - \mathcal{G}_{0A}) \tanh \frac{\omega}{2T} \quad (30)$$

The single particle Green's functions computed above for electrons coupled to reservoirs seem identical to those for electrons scattering elastically off impurities. However, the difference between the two systems will be apparent in the single particle level in the next section when an electric field is applied. In that case, while there is no steady state for electrons scattering off static impurities, the coupling to a reservoir in our model will be shown to provide an inelastic mechanism which will allow the system to reach a nonequilibrium steady state. The difference between the two systems is also apparent in equilibrium when electronic response and correlation functions are computed. For a disordered system appropriate disorder averaging gives answers which are consistent with a closed system characterized by conserved particle number. For our system the response and correlation functions (computed in Section V) will reflect the fact that the system is open since electrons can escape into the reservoir.

We now expand the $\text{Tr} \ln$ in Eq 10 about $\Delta_q \rightarrow \bar{\Delta}_q + \Delta_q$, $\Delta_{cl} \rightarrow \Delta_0 + \Delta_{cl}$, and determine $\bar{\Delta}_q, \Delta_0$ so that the resultant action is a minimum with respect to both quantum and classical fluctuations of the order-parameter. By choosing $\bar{\Delta}_q = 0$, the resultant Keldysh action is automatically minimized with respect to classical fluctuations of the order-parameter¹⁹, whereas Δ_0 will be determined by minimizing with respect to the quantum fluctuations. Thus the mean field Green's function is the matrix

$$\mathcal{G}_{mf}^{-1} = \mathcal{G}_0^{-1} + \begin{pmatrix} 0 & \Delta_0 \\ \Delta_0 & 0 \end{pmatrix} \otimes \tau_0 \quad (31)$$

where the retarded component is

$$\mathcal{G}_{Rmf}^{-1} = \begin{pmatrix} \omega - \xi_k + i\Gamma & \Delta_0 \\ \Delta_0 & \omega + \xi_k + i\Gamma \end{pmatrix} \Rightarrow \quad (32)$$

$$\mathcal{G}_{Rmf} = \frac{1}{(\omega + i\Gamma)^2 - \Delta_0^2 - \xi_k^2} \begin{pmatrix} \omega + \xi_k + i\Gamma & -\Delta_0 \\ -\Delta_0 & \omega - \xi_k + i\Gamma \end{pmatrix} \quad (33)$$

and the Keldysh component is,

$$\mathcal{G}_{Kmf} = -2i\Gamma \left[\tanh \frac{\omega}{2T} \right] \mathcal{G}_{Rmf} \mathcal{G}_{Amf} = \tanh \frac{\omega}{2T} [\mathcal{G}_{Rmf} - \mathcal{G}_{Amf}] \quad (34)$$

$$= \left[\tanh \frac{\omega}{2T} \right] \begin{pmatrix} \frac{\omega + \xi_k + i\Gamma}{(\omega + i\Gamma)^2 - \xi_k^2 - \Delta_0^2} - \frac{\omega + \xi_k - i\Gamma}{(\omega - i\Gamma)^2 - \xi_k^2 - \Delta_0^2} & -\Delta_0 \left(\frac{1}{(\omega + i\Gamma)^2 - \xi_k^2 - \Delta_0^2} - \frac{1}{(\omega - i\Gamma)^2 - \xi_k^2 - \Delta_0^2} \right) \\ -\Delta_0 \left(\frac{1}{(\omega + i\Gamma)^2 - \xi_k^2 - \Delta_0^2} - \frac{1}{(\omega - i\Gamma)^2 - \xi_k^2 - \Delta_0^2} \right) & \frac{\omega - \xi_k + i\Gamma}{(\omega + i\Gamma)^2 - \xi_k^2 - \Delta_0^2} - \frac{\omega - \xi_k - i\Gamma}{(\omega - i\Gamma)^2 - \xi_k^2 - \Delta_0^2} \end{pmatrix} \quad (35)$$

$$= \begin{pmatrix} G^K & F^K \\ \bar{F}^K & \bar{G}^K \end{pmatrix} \quad (36)$$

Δ_0 is obtained by minimizing Eq. 10 with respect to Δ_q^* which leads to the self-consistent gap equation,

$$\frac{-2i\Delta_0}{\lambda} + \text{Tr} [F_K] = 0 \quad (37)$$

The above equation, together with Eq. 36 implies the following condition for a superconducting instability

$$\frac{-2i}{\lambda} - \frac{1}{L^d} \sum_k \int_{-\Omega_{bcs}}^{\Omega_{bcs}} \frac{d\omega}{2\pi} \tanh \frac{\omega}{2T} \left[\frac{1}{(\omega + i\Gamma)^2 - \xi_k^2} - \frac{1}{(\omega - i\Gamma)^2 - \xi_k^2} \right] = 0 \quad (38)$$

For $\Gamma = 0, T \neq 0$, the above is the usual equation for the BCS mean-field transition temperature. On the other hand, at $T = 0, \Gamma \neq 0$, Eq. 38 yields a critical value of the dissipation Γ_c , above which the system becomes normal

$$\frac{1}{\nu\lambda} = \ln \frac{\Omega_{bcs}}{\Gamma_c} \quad (39)$$

where ν is the single particle density of states at the Fermi energy. The destruction of superconductivity due to large Γ can be understood in a simple way as a proximity effect. Γ measures the amount by which the states in the 2d layer broaden due to hybridization with the normal metal leads. Thus the larger is Γ the more the states in the layer acquire the property of the underlying normal metal. For an early reference on similar proximity effect induced destruction of superconductivity in a thin superconducting film deposited on a normal metal see²¹.

A second interesting feature of this model is a gapless spectrum even when there are nonzero superconducting correlations. To see this we evaluate the quasi-particle density of states per spin direction $N(\omega) = \frac{i}{2\pi} \sum_k (G^R - G^A)(k, \omega)$, and find that it is nonvanishing for $\omega < \Delta_0$. In particular at zero frequency

$$N(\omega = 0) = \frac{\nu\Gamma}{\sqrt{\Gamma^2 + \Delta_0^2}} \quad (40)$$

This appearance of gapless superconductivity is also a typical property of superconductor-metal interfaces²².

To fully understand the equilibrium properties a derivation of the superconducting action on the ordered side is presented in Appendix A. To keep the notation simple, this derivation has been done for the partition function, and the effective action for the phase fluctuations in imaginary time is found to be (Eq. A11)

$$S = \int d\tau \int d^d x \left[c_1 (\partial_\tau \theta)^2 + c_2 \left(\vec{\nabla} \theta - \frac{e}{c} \vec{A} \right)^2 \right] + S_\Sigma \quad (41)$$

where

$$S_\Sigma = \frac{g}{2\pi} \int d^d x \int d\tau \int d\tau' \left(\frac{\theta(x, \tau) - \theta(x, \tau')}{\tau - \tau'} \right)^2 \quad (42)$$

(Eq. A17). The distinguishing feature is the dissipative term S_Σ which arises because the superconducting layer is characterized by non-conserved particle number. Since the physical quantity is the voltage fluctuations $V = \partial_\tau \theta$ in the superconducting layer relative to the normal substrate, it is instructive to rewrite Eq. 42 after an integration by parts,

$$S_\Sigma = -\frac{g}{\pi} \int d^d x \int d\tau \int d\tau' \partial_\tau \theta(x, \tau) \ln |\tau - \tau'| \partial_{\tau'} \theta(x, \tau') \quad (43)$$

IV. MEAN FIELD TREATMENT OUT OF EQUILIBRIUM

We now turn to a mean-field treatment of the out of equilibrium current carrying case. A mean-field approach relies on the assumption that even in the presence of current flow the transition from the normal to the superconducting side is second order. In what follows the mean-field phase boundary in the current vs. equilibrium superconducting gap plane will be derived coming in from the disordered side. This will be followed by a discussion of the validity of mean-field.

The self-consistent mean-field equations for the current carrying case is still given by Eq. 31, however we now have to evaluate \mathcal{G}_0 which is the Green's functions for the layer electrons coupled to external reservoirs and subjected to an electric field⁸. An analytic solution may be obtained in the limit where $1/E_F \tau_{sc} \ll 1$ and $\left(\frac{T_{eff}}{E_F}\right) T_{eff} \tau_{sc} \ll 1$, where $\tau_{sc} = 1/(2\Gamma)$ is the typical escape time into the reservoir, and $T_{eff} = eEv_F \tau_{sc}$ is an effective temperature that characterizes the steady state distribution function of the layer electrons. The details are presented in Appendix B. Within this approximation the mean-field green's functions $\mathcal{G}_{mf}^{R,A}$ remain unchanged and are still given by Eq. 33, while \mathcal{G}_{mf}^K changes due to a nonequilibrium electronic distribution function. In what follows, both in the mean-field treatment of this section, and the study of fluctuations in the next section and in Appendix C, we will make the additional assumption that $T_{eff} \tau_{sc} \ll 1$.

Using the expression for the distribution function in Eq. B15, B16, B17, the self-consistent gap equation Eq 37 becomes,

$$\frac{2i}{\nu\lambda} = - \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_0^{\Omega_{bcs}} \frac{d\omega}{\pi} \left(1 - e^{\frac{-\omega}{T_{eff} |\cos \theta|}} \right) \int_{-\infty}^{\infty} d\xi \left[\frac{1}{(\omega + i\Gamma)^2 - \xi^2 - \Delta_0^2} - \frac{1}{(\omega - i\Gamma)^2 - \xi^2 - \Delta_0^2} \right] \quad (44)$$

We may approximate $\int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_0^{\Omega_{bcs}} \frac{d\omega}{\pi} \left(1 - e^{\frac{-\omega}{T_{eff} |\cos \theta|}} \right) \simeq \int_{T_{eff}}^{\Omega_{bcs}} \frac{d\omega}{\pi}$. Defining $T_{c,eff}$ as the critical value of the current induced effective temperature for which $\Delta_0 = 0$ in Eq. 44, and relating $\frac{1}{\nu\lambda}$ to the gap Δ_{eq} in equilibrium we find

$$\frac{2}{\pi} \int_0^{\Omega_{bcs}} \frac{d\xi}{\sqrt{\xi^2 + \Delta_{eq}^2}} \arctan \frac{\sqrt{\xi^2 + \Delta_{eq}^2}}{\Gamma} = \ln \frac{\Omega_{bcs}}{\sqrt{T_{c,eff}^2 + \Gamma^2}} \quad (45)$$

An approximate solution to the above equation is $T_{c,eff} \simeq \Delta_{eq}/\sqrt{2}$. Note that this current induced loss of order is a heating effect arising due to a highly broadened electron distribution function, and is not the same as the Landau-criterion for the critical current for breakdown of superfluidity.

We now turn to the discussion of the validity of the above mean-field treatment. Firstly, within mean-field T_{eff} arises solely out of noise due to normal electron current, and leaves out the fact that current due to superconducting fluctuations also contribute to noise that can modify T_{eff} . Secondly, other scenarios for a current induced transition are possible. For example, on the superconducting side there is no dissipative current, so that $T_{eff} = 0$. In this case it is possible to have a supercurrent induced first order transition from a superconducting state to a normal or resistive state as discussed in^{23,24}. Whether the actual transition is a second order heating effect as predicted by the mean-field treatment on the disordered side or a first order transition depends on whether the critical current for the first order transition is larger or smaller than the current corresponding to $T_{c,eff}$. In general this is a complex question that we do not address further.

Instead, in what follows we will derive a nonequilibrium Ginzburg-Landau theory for the superconducting fluctuations on the normal side where most of the current is due to normal electrons so that there is a well defined T_{eff} . We will then use this to study how the gap and the current due to superconducting fluctuations scale due to electric field close to the quantum critical point and outside the fluctuation dominated Ginzburg regime.

V. FLUCTUATION ABOUT NONEQUILIBRIUM DISORDERED STATE: DERIVATION OF THE KELDYSH GINZBURG-LANDAU FUNCTIONAL

We now turn to the discussion of fluctuations about the mean-field disordered state ($\Delta_0 = 0$ in Eq. 31). In doing so we will also highlight the difference between how the electric field affects magnetic-fluctuations⁸ and superconducting fluctuations.

Expanding the $Tr \ln$ in Eq. 10 in the usual way $Tr \ln \mathcal{G}^{-1} = Tr \ln \mathcal{G}_0^{-1} + Tr \mathcal{G}_0 \hat{\Delta} - \frac{1}{2} Tr \mathcal{G}_0 \hat{\Delta} \mathcal{G}_0 \hat{\Delta} + \dots$ where

$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta_{cl} \\ \Delta_{cl}^* & 0 \end{pmatrix} \otimes \tau_0 + \begin{pmatrix} 0 & \Delta_q \\ \Delta_q^* & 0 \end{pmatrix} \otimes \tau_x \quad (46)$$

one obtains an effective action

$$Z_K = \int \mathcal{D} [\Delta_{q,cl}, \Delta_{q,cl}^*] e^{-iS_K^2 - iS_K^3 - iS_K^4 + \dots} \quad (47)$$

with

$$S_K^2 = Tr \begin{pmatrix} \Delta_q^* & \Delta_{cl}^* \end{pmatrix} \begin{pmatrix} \Pi_{G\bar{G}}^K & \frac{2}{\lambda} + \Pi_{G\bar{G}}^R \\ \frac{2}{\lambda} + \Pi_{G\bar{G}}^A & 0 \end{pmatrix} \begin{pmatrix} \Delta_q \\ \Delta_{cl} \end{pmatrix} \quad (48)$$

In position and time space $1 = x, t$ and $2 = x', t'$

$$\Pi_{G\bar{G}}^R(1, 2) = -i [G_{0R}(1, 2) \bar{G}_{0K}(2, 1) + G_{0K}(1, 2) \bar{G}_{0A}(2, 1)] \quad (49)$$

$$\Pi_{G\bar{G}}^K(1, 2) = -i [G_{0K}(1, 2) \bar{G}_{0K}(2, 1) + G_{0R}(1, 2) \bar{G}_{0A}(2, 1) + G_{0A}(1, 2) \bar{G}_{0R}(2, 1)] \quad (50)$$

Note that on the disordered side, terms cubic order in the superconducting fluctuations are absent ($S_K^3 = 0$), while S_K^4 has the form⁵

$$S_K^4 = \sum_{i=1\dots 4} u_i \Delta_q^{*i} \Delta_{cl}^{4-i} + c.c. \quad (51)$$

We will treat S_K^4 only within a one-loop mean-field approximation. For this only the coupling constant u_1 will play a role.

Since $\bar{G}_R(1, 2) = -G_A(2, 1) \Rightarrow \bar{G}_R(\vec{k}, \omega) = -G_A(-\vec{k}, -\omega)$, $\bar{G}_K(1, 2) = -G_K(2, 1) \Rightarrow \bar{G}_K(\vec{k}, \omega) = -G_K(-\vec{k}, -\omega)$, we may write

$$Tr[\Delta_q^* \Pi_{G\bar{G}}^R \Delta_{cl}] = iTr\Delta_q^*(1) [G_{0R}(1, 2)G_{0K}(1, 2) + G_{0K}(1, 2)G_{0R}(1, 2)] \Delta_{cl}(2) \quad (52)$$

$$Tr[\Delta_q^* \Pi_{G\bar{G}}^K \Delta_q] = iTr\Delta_q^*(1) [G_{0K}(1, 2)G_{0K}(1, 2) + G_{0R}(1, 2)G_{0R}(1, 2) + G_{0A}(1, 2)G_{0A}(1, 2)] \Delta_q(2) \quad (53)$$

The coefficients Π depend explicitly on the electric field. Since we use the Gauge $\vec{A} = -c\vec{E}t$, it is most convenient to go into momentum and time space,

$$Tr[\Delta_q^* \Pi_{G\bar{G}}^R \Delta_{cl}] = iTr\Delta_q^*(-\vec{q}, t_1) [G_{0R}(\vec{p} + \vec{q}; t_1, t_2)G_{0K}(-\vec{p}; t_1, t_2) + G_{0K}(\vec{p} + \vec{q}; t_1, t_2)G_{0R}(-\vec{p}; t_1, t_2)] \Delta_{cl}(\vec{q}, t_2) \quad (54)$$

Note that for magnetic fluctuations, the above expression would have had the form⁸

$$Tr[m_q \Pi_{G\bar{G}}^R m_{cl}] = iTrm_q^*(-\vec{q}, t_1) [G_{0R}(\vec{p} + \vec{q}; t_1, t_2)G_{0K}(\vec{p}; t_2, t_1) + G_{0K}(\vec{p} + \vec{q}; t_1, t_2)G_{0A}(\vec{p}; t_2, t_1)] m_{cl}(\vec{q}, t_2) \quad (55)$$

As shown in Appendix B, if the single particle Green's functions are written in terms of the canonical momentum $\vec{k} = \vec{p} + e\vec{E}T$ where $T = \frac{t_1+t_2}{2}$, they become time translationally invariant. Thus for superconducting fluctuations one may write Eq. 54 as

$$Tr[\Delta_q^* \Pi_{G\bar{G}}^R \Delta_{cl}] = iTr\Delta_q^*(-\vec{q}, t_1) \{G_{0R}(\vec{p} + \vec{q} + e\vec{E}T; t_1 - t_2)G_{0K}(-\vec{p} + e\vec{E}T; t_1 - t_2) + G_{0K}(\vec{p} + \vec{q} + e\vec{E}T; t_1 - t_2)G_{0R}(-\vec{p} + e\vec{E}T; t_1 - t_2)\} \Delta_{cl}(q, t_2) \quad (56)$$

In terms of the canonical momentum $k = p + eET$, the above becomes

$$Tr[\Delta_q^* \Pi_{G\bar{G}}^R \Delta_{cl}] = iTr\Delta_q^*(-\vec{q}, t_1) \{G_{0R}(\vec{k} + \vec{q}; t_1 - t_2)G_{0K}(-\vec{k} + 2e\vec{E}T; t_1 - t_2) + G_{0K}(\vec{k} + \vec{q}; t_1 - t_2)G_{0R}(-\vec{k} + 2e\vec{E}T; t_1 - t_2)\} \Delta_{cl}(q, t_2) \quad (57)$$

or shifting variables $\vec{k} \rightarrow \vec{k} + 2e\vec{E}T$ one may write

$$Tr[\Delta_q^* \Pi_{G\bar{G}}^R \Delta_{cl}] = iTr\Delta_q^*(-\vec{q}, t_1) \{G_{0R}(\vec{k} + \vec{q} + 2e\vec{E}T; t_1 - t_2)G_{0K}(-\vec{k}; t_1 - t_2) + G_{0K}(\vec{k} + \vec{q} + 2e\vec{E}T; t_1 - t_2)G_{0R}(-\vec{k}; t_1 - t_2)\} \Delta_{cl}(q, t_2) \quad (58)$$

Following the same steps for magnetic fluctuations we get

$$Tr[m_q \Pi_{G\bar{G}}^R m_{cl}] = iTrm_q^*(-\vec{q}, t_1) [G_{0R}(\vec{p} + \vec{q} + e\vec{E}T; t_1 - t_2)G_{0K}(\vec{p} + e\vec{E}T, t_2 - t_1) + G_{0K}(\vec{p} + \vec{q} + e\vec{E}T; t_1 - t_2)G_{0A}(\vec{p} + e\vec{E}T; t_2 - t_1)] m_{cl}(\vec{q}, t_2) \quad (59)$$

Rewriting the above in terms of the canonical momentum $\vec{k} = \vec{p} + e\vec{E}T$, one finds,

$$Tr[m_q \Pi_{G\bar{G}}^R m_{cl}] = iTrm_q^*(-\vec{q}, t_1) [G_{0R}(\vec{k} + \vec{q}; t_1 - t_2)G_{0K}(\vec{k}, t_2 - t_1) + G_{0K}(\vec{k} + \vec{q}; t_1 - t_2)G_{0A}(\vec{k}; t_2 - t_1)] m_{cl}(\vec{q}, t_2) \quad (60)$$

Thus Eq. 60 and 58 highlight the difference between the coupling of the electric field to the magnetic and superconducting order parameters. In Eq. 60, all dependence of the electric field is via the modification of the Green's functions $G^{R,K}$ at steady state, and there is no direct coupling between the electric field and the order-parameter. On the other hand Eq. 58 depends on the combination $(\vec{q} + 2e\vec{E}T)$ which is the usual minimal coupling of the charged superconducting fluctuation and an external electric field.

Thus to summarize, upto quadratic order, the Keldysh action for superconducting fluctuations in the presence of an electric field may be written as

$$S_K^2 = \int dt_1 \int dt_2 \sum_{\vec{q}} \left(\begin{array}{cc} \Delta_q^*(-\vec{q}, t_1) & \Delta_{cl}^*(-\vec{q}, t_1) \end{array} \right) \left(\begin{array}{cc} \Pi_{G\bar{G}}^K(\vec{q} + 2e\vec{E}T, t_1 - t_2) & \frac{2}{\lambda}\delta(t_1 - t_2) + \Pi_{G\bar{G}}^R(\vec{q} + 2e\vec{E}T, t_1 - t_2) \\ \frac{2}{\lambda}\delta(t_1 - t_2) + \Pi_{G\bar{G}}^A(\vec{q} + 2e\vec{E}T, t_1 - t_2) & 0 \end{array} \right) \left(\begin{array}{c} \Delta_q(\vec{q}, t_2) \\ \Delta_{cl}(\vec{q}, t_2) \end{array} \right) \quad (61)$$

where $T = \frac{t_1+t_2}{2}$. We now discuss the coefficients $\Pi^{R,A,K}$ and highlight the appearance of new current dependent terms that were missed in previous phenomenological treatments.

We expand the Π bubbles in powers of $(\vec{q} + 2e\vec{E}T)$ to obtain,

$$\Pi^R(\vec{q} + 2e\vec{E}T; t_1 - t_2) = \left[\Pi_R^0(t_1 - t_2) + \vec{E} \cdot (\vec{q} + 2e\vec{E}T) \Pi_1^R(t_1 - t_2) + (\vec{q} + 2e\vec{E}T)^2 \Pi_2^R(t_1 - t_2) + \dots \right] \quad (62)$$

$$\Pi^K(\vec{q} + 2e\vec{E}T; t_1 - t_2) = \Pi_0^K(t_1 - t_2) + \mathcal{O}((\vec{q} + 2e\vec{E}T)^2) \quad (63)$$

It is convenient to Fourier transform the above expressions so that

$$\Pi^R(\vec{q} + 2e\vec{E}T; t_1 - t_2) = \int \frac{d\Omega}{2\pi} e^{-i\Omega(t_1-t_2)} \left[\tilde{\Pi}_0^R(\Omega) + \vec{E} \cdot (\vec{q} + 2e\vec{E}T) \tilde{\Pi}_1^R(\Omega) + (\vec{q} + 2e\vec{E}T)^2 \tilde{\Pi}_2^R(\Omega) \right] \quad (64)$$

Each of $\tilde{\Pi}_{0,1}^R(\Omega)$ can be evaluated as a power series in Ω (see Appendix C for details). Keeping terms to $\mathcal{O}(\Omega, (\vec{q} + 2e\vec{E}T)^2)$ one obtains,

$$\delta(t_1 - t_2) + \frac{\lambda}{2} \Pi^R(\vec{q} + 2e\vec{E}T; t_1 - t_2) = \delta(t_1 - t_2) \left[\eta \left(\frac{\partial}{\partial t_1} - i\tau_{sc} \frac{e\vec{E} \cdot (\vec{q} + 2e\vec{E}T)}{m} \right) + \delta + \gamma (\vec{q} + 2e\vec{E}T)^2 + \dots \right] \quad (65)$$

where, as derived in Appendix C

$$\eta = \nu\lambda\tau_{sc} \quad (66)$$

$$\delta = 1 + \frac{\lambda}{2} Re \left[\tilde{\Pi}^R(0, 0) \right] \quad (67)$$

$$\gamma = \lambda\nu \frac{\mu}{4m\Gamma^2} \quad (68)$$

The first term on the r.h.s of Eq. 65 is the overdamped dynamics associated with non-conservation of particle number, while the second term is of the form $\vec{v}_D \cdot (\vec{q} + 2e\vec{A}t)$ and represents current induced drift at velocity

$$v_D = \frac{\tau_{sc} e E}{m} \quad (69)$$

The difference with⁹ is the appearance of the above drift term, along with a change in the noise properties of the reservoir (represented by Π^K) due to current flow. In particular, we find the following electric-field dependence of Π^K in 2d (see Appendix C for details)

$$\Pi^K(\Omega) = -4i\nu\tau_{sc} \left[|\Omega| + T_{eff} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} |\cos \phi| e^{-\frac{|\Omega|}{T_{eff} |\cos \phi|}} \right] \quad (70)$$

as opposed to a current independent $\Pi^K(\Omega) \propto |\Omega|$, in the model studied in⁹.

Note that for a 1d system, the structure of $\Pi^{R,A}$ remain the same as in 2d, while Π^K acquires the form in Eq. C6. Qualitatively it has the same structure as Eq. 70 in that $\Pi_{1d}^K \propto |\Omega|$ when $|\Omega| > T_{eff}$, and $\Pi_{1d}^K \propto T_{eff}$ when $\Omega = 0$.

We now turn to the evaluation of the gap-equation and the current due to superconducting fluctuations to see what role these new terms due to current induced noise and drift play.

VI. EVALUATION OF SELF-CONSISTENT GAP AND CURRENT DUE TO SUPERCONDUCTING FLUCTUATIONS

In order to derive the self-consistent gap equation and the fluctuation conductivity, as in⁹ we will work to quadratic order (Eq. 61), treating the quartic term in superconducting fluctuations (Eq. 51) within a one-loop mean-field approximation.

We may define the retarded, advanced and Keldysh component of the Green's functions for the superconducting fluctuations as follows

$$D^R(1, 2) = -i\theta(t_1 - t_2) \langle [\Delta(1), \Delta^*(2)] \rangle = -i \langle \Delta_q(1) \Delta_{cl}^*(2) \rangle \quad (71)$$

$$D^A(1, 2) = i\theta(t_2 - t_1) \langle [\Delta(1), \Delta^*(2)] \rangle = -i \langle \Delta_{cl}(1) \Delta_q^*(2) \rangle \quad (72)$$

$$D^K(1, 2) = -i \langle \{ \Delta(1), \Delta^*(2) \} \rangle = -i \langle \Delta_{cl}(1) \Delta_{cl}^*(2) \rangle \quad (73)$$

From Eqns 61, 65 and 70, the equation of motion obeyed by the above Green's functions are

$$D^K = -D^R \Pi^K D^A \quad (74)$$

where

$$\left[\eta \left(\frac{\partial}{\partial t_1} - i\vec{v}_D \cdot (\vec{q} + 2e\vec{E}t_1) \right) + \delta + \gamma (\vec{q} + 2e\vec{E}t_1)^2 \right] D^R(\vec{q}; t_1, t_2) = -\delta(t_1, t_2) \quad (75)$$

The above equation corresponds to overdamped dynamics and may be solved easily,

$$D^R(\vec{q}; t_1, t_2) = -\theta(t_1 - t_2) \frac{1}{\eta} \exp \left(-\frac{1}{\eta} \int_{t_2}^{t_1} d\tau \left[\epsilon_q(\tau) - i\eta \vec{v}_D \cdot (\vec{q} + 2e\vec{E}\tau) \right] \right) \quad (76)$$

$$D^A(q; t_1, t_2) = -\theta(t_2 - t_1) \frac{1}{\eta} \exp \left(\frac{1}{\eta} \int_{t_2}^{t_1} d\tau \left[\epsilon_q(\tau) + i\eta \vec{v}_D \cdot (\vec{q} + 2e\vec{E}\tau) \right] \right) \quad (77)$$

where

$$\epsilon_q(\tau) = \delta + \gamma (\vec{q} + 2e\vec{E}\tau)^2 \quad (78)$$

A. Self-consistent gap equation

The self-consistent gap equation is

$$\delta = \delta_0 + u_1 \langle |\Delta_{cl}|^2 \rangle \quad (79)$$

where $u_1 \sim \frac{\gamma}{\eta} \nu \lambda \tau_{sc}^2$ and

$$\begin{aligned} \langle |\Delta_{cl}|^2 \rangle &= iD^K(x, t; x, t) = -i \int d2d3 D^R(1, 2) \Pi^K(2, 3) D^A(3, 1) \\ &= -i \int \frac{d^2q}{(2\pi)^2} \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 D^R(q; t, t_1) \Pi^K(t_1, t_2) D^A(q; t_2, t) \end{aligned} \quad (80)$$

In Fourier space $\Pi^K(t_1, t_2) = \int \frac{d\Omega}{2\pi} e^{-i\Omega(t_1-t_2)} \Pi^K(\Omega)$ which together with Eqns 76, 77, 70 give

$$\begin{aligned} \langle |\Delta_{cl}|^2 \rangle &= -i \int \frac{d^2q}{(2\pi)^2} \int \frac{d\Omega}{2\pi} \Pi^K(\Omega) \int_{-\infty}^t dt_1 \int_{-\infty}^t dt_2 \frac{1}{\eta^2} e^{-i\Omega(t_1-t_2)} e^{-\frac{1}{\eta} \int_{t_1}^t d\tau_1 [\delta + \gamma (\vec{q} + 2e\vec{E}\tau_1)^2 - i\eta \vec{v}_D \cdot (\vec{q} + 2e\vec{E}\tau_1)]} \\ &\quad e^{-\frac{1}{\eta} \int_{t_2}^t d\tau_2 [\delta + \gamma (\vec{q} + 2e\vec{E}\tau_2)^2 + i\eta \vec{v}_D \cdot (\vec{q} + 2e\vec{E}\tau_2)]} \end{aligned} \quad (81)$$

Changing variables to the canonical momentum $\vec{k} = \vec{q} + 2e\vec{E}t$, the explicit dependence on t goes away, and one obtains

$$\begin{aligned} \langle |\Delta_{cl}|^2 \rangle &= -i \int \frac{d^2k}{(2\pi)^2} \int \frac{d\Omega}{2\pi} \Pi^K(\Omega) \frac{1}{\eta^2} \int_0^\infty dx \int_0^\infty dy \\ &\quad e^{i\Omega(x-y) + i\vec{v}_D \cdot \vec{k}(x-y) - i\vec{v}_D \cdot e\vec{E}(x^2-y^2)} e^{-\frac{1}{\eta} (\delta + \gamma k^2)(x+y)} e^{-\frac{\gamma}{3\eta} (2eE)^2 (x^3+y^3)} e^{\frac{\gamma}{\eta} \vec{k} \cdot 2e\vec{E}(x^2+y^2)} \end{aligned} \quad (82)$$

It is convenient to perform the momentum integrals, which gives,

$$\begin{aligned} \langle |\Delta_{cl}|^2 \rangle &= \frac{-i}{4\pi^2} \frac{\eta\pi}{\gamma} \int \frac{d\Omega}{2\pi} \Pi^K(\Omega) \frac{1}{\eta^2} \int_0^\infty dx \int_0^\infty dy e^{i\Omega(x-y) - i\vec{v}_D \cdot e\vec{E}(x^2-y^2)} \frac{1}{x+y} e^{-\frac{\delta}{\eta}(x+y)} \\ &\quad e^{-\frac{4\gamma}{3\eta} (eE)^2 (x^3+y^3)} e^{\frac{\gamma}{\eta} (eE)^2 \frac{(x^2+y^2)^2}{x+y}} \left(1 + i \frac{v_D}{eE} \left(\frac{\eta}{2\gamma} \right) \frac{x-y}{x^2+y^2} \right)^2 \end{aligned} \quad (83)$$

After this the manipulations are similar to⁹. It is convenient to change variables to $u = x + y, v = x - y$, so that $\int_0^\infty dx \int_0^\infty dy = \frac{1}{2} \int_0^\infty du \int_{-u}^u dv$ giving

$$\begin{aligned} \langle |\Delta_{cl}|^2 \rangle &= \frac{-i}{4\pi^2} \frac{\eta\pi}{\gamma} \int \frac{d\Omega}{2\pi} \Pi^K(\Omega) \frac{1}{2\eta^2} \int_0^\infty \frac{du}{u} \int_{-u}^u dv \\ &\quad e^{i\Omega v - i\vec{v}_D \cdot e\vec{E}uv} e^{-\frac{\delta}{\eta}u} e^{-\frac{\gamma}{3\eta} (eE)^2 u(u^2+3v^2)} e^{\frac{\gamma}{4\eta} (eE)^2 \frac{(u^2+v^2)^2}{u}} \left(1 + \frac{iv_D \eta}{eE\gamma} \frac{v}{u^2+v^2} \right)^2 \end{aligned} \quad (84)$$

Now we approximate the expression for Π^K in Eq. 70 as $\Pi^K(\Omega) \simeq 2i\eta \left[|\Omega| + \frac{2\bar{T}_{eff}}{\pi} \theta(|\Omega| - T_{eff}) \right]$. We also define dimensionless variables $\bar{\Omega} = \Omega\eta, u/\eta \rightarrow u, v/\eta \rightarrow v$

$$\bar{T}_{eff} = T_{eff}\eta \quad (85)$$

$$\bar{E} = eE\eta\sqrt{\gamma} \quad (86)$$

in terms of which

$$\begin{aligned} \langle |\Delta_{cl}|^2 \rangle &= \frac{1}{2\pi\gamma\eta} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} \left[|\bar{\Omega}| + \frac{2\bar{T}_{eff}}{\pi} \theta(|\bar{\Omega}| - T_{eff}) \right] \int_0^{\infty} \frac{du}{u} \int_0^u dv \cos \left(\bar{\Omega}v - \frac{\tau_{sc}}{\gamma m} \bar{E}^2 uv + \frac{\tau_{sc}}{2\gamma m} \bar{E}^2 \frac{v}{u} \right) \\ &e^{-\delta u} e^{-\frac{\bar{E}^2}{3} u(u^2+3v^2)} e^{-\frac{\bar{E}^2}{4} \frac{(u^2+v^2)^2}{u}} \left(1 - \frac{\tau_{sc}^2}{\gamma^2 m^2} \frac{u^2}{(u^2+v^2)^2} \right) \end{aligned} \quad (87)$$

Changing variables to $v \rightarrow v/u$, one gets

$$\begin{aligned} \langle |\Delta_{cl}|^2 \rangle &= \frac{1}{2\pi\gamma\eta} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} \left[|\bar{\Omega}| + \frac{2\bar{T}_{eff}}{\pi} \theta(|\bar{\Omega}| - T_{eff}) \right] \int_0^{\infty} du \int_0^1 dv \cos \left(\bar{\Omega}uv - \frac{\tau_{sc}}{\gamma m} \bar{E}^2 u^2 + \frac{\tau_{sc}}{2\gamma m} \bar{E}^2 v \right) \\ &e^{-\delta u} e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} e^{-\frac{\bar{E}^2}{4} \frac{\tau_{sc}^2}{\gamma^2 m^2} uv^2 (1+v^2)} \end{aligned} \quad (88)$$

It is now straightforward to see the role played by current drift. This term always arises in the combination $\frac{\tau_{sc}}{\gamma m} \bar{E}^2$. Using Eq. 68, one finds it to be $\mathcal{O}\left(\frac{\bar{E}^2}{\mu\tau_{sc}}\right)$. As we shall show, the electric field scaling due to the current noise term is $\mathcal{O}(\bar{E}^2)$ in the quantum-disordered regime. Thus the drift gives corrections to this result by an amount which is smaller by a factor of $\frac{1}{E_F\tau_{sc}} \ll 1$ ($E_F = \mu$). Therefore in what follows we will drop the drift term from further analysis.

Substituting Eq. 88 in Eq. 79, and adding subtracting terms one gets the following self-consistent gap equation

$$\delta = \delta_0 + \frac{u_1}{2\pi\gamma\eta} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} |\bar{\Omega}| \int_0^{\infty} du \int_0^1 dv \cos(\bar{\Omega}uv) e^{-\delta u} \quad (89)$$

$$\begin{aligned} &+ \frac{u_1}{2\pi\gamma\eta} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} |\bar{\Omega}| \int_0^{\infty} du \int_0^1 dv \cos(\bar{\Omega}uv) e^{-\delta u} \left[e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} - e^{-\frac{\bar{E}^2 u^3}{12}} + e^{-\frac{\bar{E}^2 u^3}{12}} - 1 \right] \\ &+ \frac{u_1}{2\pi\gamma\eta} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} \left[\frac{2\bar{T}_{eff}}{\pi} \theta(|\bar{\Omega}| - T_{eff}) \right] \int_0^{\infty} du \int_0^1 dv \cos(\bar{\Omega}uv) e^{-\delta u} e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} \end{aligned} \quad (90)$$

We introduce a frequency cutoff Λ in the first term in the above equation, and perform the frequency integral to obtain,

$$\delta \left[1 + \frac{u_1}{2\pi^2\gamma\eta} \ln \frac{1}{\delta} \right] = \left[\delta_0 + \frac{u_1}{2\pi^2\gamma\eta} \left(\Lambda \frac{\pi}{2} - \delta \ln \Lambda \right) \right] \quad (91)$$

$$\begin{aligned} &+ \frac{u_1}{2\pi\gamma\eta} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} |\bar{\Omega}| \int_0^{\infty} du \int_0^1 dv \cos(\bar{\Omega}uv) e^{-\delta u} \left[e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} - e^{-\frac{\bar{E}^2 u^3}{12}} + e^{-\frac{\bar{E}^2 u^3}{12}} - 1 \right] \\ &+ \frac{u_1}{2\pi\gamma\eta} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} \left[\frac{2\bar{T}_{eff}}{\pi} \theta(|\bar{\Omega}| - T_{eff}) \right] \int_0^{\infty} du \int_0^1 dv \cos(\bar{\Omega}uv) e^{-\delta u} e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} \end{aligned} \quad (92)$$

The remaining frequency integrals above are performed by introducing a cutoff $e^{-\Lambda^{-1}|\Omega|}$ in the argument, and then setting $\Lambda^{-1} = 0$. For example, one integral evaluates to $Lt_{\Lambda^{-1} \rightarrow 0} \int_0^{\infty} \Omega \cos(\Omega uv) e^{-\Omega/\Lambda} = \frac{\Lambda^{-1} - u^2 v^2}{(\Lambda^{-2} + u^2 v^2)^2} = \frac{-1}{u^2 v^2}$, while another is $Lt_{\Lambda^{-1} \rightarrow 0} \int_0^{\infty} d\Omega \sin(\Omega u) e^{-\Omega/\Lambda} = \frac{1}{u}$.

In addition, by defining, $\delta^R = \frac{\delta_0}{\frac{u_1}{2\pi^2\gamma\eta}} + [\Lambda \frac{\pi}{2} - \delta \ln \Lambda]$ as the renormalized distance from the QCP, and by using $\ln 1/\delta \gg 1$, the self-consistent gap equation becomes,

$$\begin{aligned} \delta \ln \frac{1}{\delta} &= \delta^R - \int_0^{\infty} du \int_0^1 dv \frac{1}{u^2 v^2} e^{-\delta u} \left[e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} - e^{-\frac{\bar{E}^2 u^3}{12}} \right] \\ &+ \int_0^{\infty} du \frac{1}{u^2} e^{-\delta u} \left[e^{-\frac{\bar{E}^2 u^3}{12}} - 1 \right] + \left(\frac{2\bar{T}_{eff}}{\pi} \right) \int_0^{\infty} du \int_0^1 dv \frac{1}{uv} \sin(\bar{T}_{eff} uv) e^{-\delta u} e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} \end{aligned} \quad (93)$$

The first three terms on the r.h.s was derived in⁹, whereas the last term is new and arises due to current noise and reflects the modification of the underlying electron distribution function.

We discuss the solution of the gap equation in two regimes

A. Quantum Disordered Regime, $\delta^R \gg \bar{E}$. Here Eq. 93 can be perturbatively expanded in powers of \bar{E} to give

$$\delta \simeq \frac{\delta^R}{\ln \frac{1}{\delta^R}} + \frac{\bar{E}^2}{3(\delta^R / \ln \frac{1}{\delta^R})^2} + \frac{2}{\pi} \frac{\bar{T}_{eff}^2}{(\delta^R / \ln \frac{1}{\delta^R})} \quad (94)$$

While the first two terms in Eq. 94 were derived in⁹, the last term is the correction due to current noise which essentially acts as an effective temperature. As discussed before, current drift will correct this result by a factor of $\mathcal{O}(1/E_F \tau_{sc})$.

B. Quantum Critical Regime, $\delta^R \ll \bar{E}$. Here one may set $e^{\delta u} = 1$ in Eq. 93, which in terms of a rescaled variable $\bar{u} = uE^{2/3}$, may be written as

$$\begin{aligned} \delta \ln \frac{1}{\delta} = \bar{E}^{2/3} & \left[\int_0^\infty d\bar{u} \frac{1}{\bar{u}^2} \left(e^{-\frac{\bar{u}^3}{12}} - 1 \right) - \int_0^\infty d\bar{u} \int_0^1 dv \frac{1}{\bar{u}^2 v^2} \left(e^{-\frac{\bar{u}^3}{12}(1+6v^2-3v^4)} - e^{-\frac{\bar{u}^3}{12}} \right) \right] \\ & + \left(\frac{2\bar{T}_{eff}^2}{\pi \bar{E}^{2/3}} \right) \int_0^\infty d\bar{u} \int_0^1 dv e^{-\frac{\bar{u}^3}{12}(1+6v^2-3v^4)} \end{aligned} \quad (95)$$

Defining the following functions

$$\mathcal{Y} = \frac{1}{3^{1/3} 2^{4/3}} \Gamma\left(\frac{2}{3}\right) \int_0^1 dv \frac{\left[(1+6v^2-3v^4)^{1/3} - (1+v^2) \right]}{v^2} = 0.1165 \quad (96)$$

$$\mathcal{Y}' = \int_0^\infty d\bar{u} \int_0^1 dv e^{-\frac{\bar{u}^3}{12}(1+6v^2-3v^4)} = 1.603 \quad (97)$$

we find,

$$\delta \simeq \frac{(2\bar{E})^{2/3} \mathcal{Y}}{\ln \frac{1}{(2\bar{E})^{2/3}}} \left[1 + 5.52 \frac{\bar{T}_{eff}^2}{\bar{E}^{4/3}} \right] \quad (98)$$

Again the first term was derived in⁹, while the second term above is the correction arising due to the modification of the distribution function of the underlying electrons. Using the definitions Eq. 85, 86 and the expressions for η and γ in Eqns. 66, 68, $\frac{\bar{T}_{eff}^2}{\bar{E}^{4/3}} \sim (T_{eff} \tau_{sc})^{2/3} \ll 1$. Thus this term only gives rise to subleading corrections within the model presented here where τ_{sc} is independent of the electric-field. In the conclusions we discuss the case of systems where τ_{sc} may have a strong electric-field dependence, and can in particular diverge as $E \rightarrow 0$. In this case it may be possible for the second term to dominate over the first.

B. Expression for the current due to superconducting fluctuations

We now turn to the evaluation of current due to superconducting fluctuations. The expression for the current is given by

$$\vec{J} = \frac{\delta Z_K}{\delta \vec{A}} = \frac{2e}{\hbar} \gamma \int \frac{d^2 q}{(2\pi)^2} \left(\vec{q} + 2e\vec{E}t \right) iD^K(q; t, t) \quad (99)$$

Changing variables to the canonical momentum $\vec{k} = \vec{q} + 2e\vec{E}t$, and using Eq. 74, we obtain the expression

$$\vec{J} = -i \frac{2e}{\hbar} \gamma \int \frac{d^2 k}{(2\pi)^2} \vec{k} \int \frac{d\Omega}{2\pi} \Pi^K(\Omega) \frac{1}{\eta^2} \int_0^\infty dx \int_0^\infty dy e^{i\Omega(x-y)} e^{-\frac{1}{\eta}(\delta + \gamma k^2)(x+y)} e^{-\frac{\gamma}{3\eta}(2eE)^2(x^3+y^3)} e^{\frac{\gamma}{\eta} \vec{k} \cdot 2e\vec{E}(x^2+y^2)} \quad (100)$$

Performing the momentum integral, one gets

$$J = \frac{-i}{4\pi^2} \frac{\eta\pi}{\gamma} \left(\frac{2e}{\hbar} \right) \gamma eE \int \frac{d\Omega}{2\pi} \Pi^K(\Omega) \frac{1}{\eta^2} \int_0^\infty dx \int_0^\infty dy e^{i\Omega(x-y)} \frac{x^2 + y^2}{(x+y)^2} e^{-\frac{\delta}{\eta}(x+y)} e^{-\frac{4\gamma}{3\eta}(eE)^2(x^3+y^3)} e^{\frac{\gamma}{\eta}(eE)^2 \frac{(x^2+y^2)^2}{x+y}} \quad (101)$$

As before we perform a change of variables to previously defined dimensionless variables to obtain,

$$J = \frac{2e^2}{\hbar} \frac{E}{2\pi} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} \left[|\bar{\Omega}| + \frac{2\bar{T}_{eff}}{\pi} \theta(|\Omega| - T_{eff}) \right] \int_0^{\infty} du \int_0^1 dv \cos(\bar{\Omega}uv) \frac{u}{2} (1+v^2) e^{-\delta u} e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} \quad (102)$$

Adding and subtracting terms in Eq 102,

$$J = \frac{2e^2}{\hbar} \frac{E}{2\pi} \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} |\bar{\Omega}| \int_0^{\infty} du \int_0^1 dv \cos(\bar{\Omega}uv) \frac{u}{2} (1+v^2) e^{-\delta u} \left[e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} - e^{-\frac{\bar{E}^2 u^3}{12}} + e^{-\frac{\bar{E}^2 u^3}{12}} \right] \quad (103)$$

$$+ \frac{2e^2}{\hbar} \frac{E}{2\pi} \left(\frac{2\bar{T}_{eff}}{\pi} \right) \int_{-\infty}^{\infty} \frac{d\bar{\Omega}}{2\pi} [\theta(|\Omega| - T_{eff})] \int_0^{\infty} du \int_0^1 dv \cos(\bar{\Omega}uv) \frac{u}{2} (1+v^2) e^{-\delta u} e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)}$$

One of the integrals that may be performed is $Lt_{\Lambda^{-1} \rightarrow 0} \int_0^{\infty} d\Omega \Omega e^{-\Omega/\Lambda} \int_0^1 dv \cos(\Omega uv) (1+v^2) = \int_0^{\infty} d\Omega \Omega e^{-\Omega/\Lambda} \frac{2}{u^3 \Omega^3} [u\Omega \cos(u\Omega) + (-1+u^2\Omega^2) \sin(u\Omega)] = Lt_{\Lambda^{-1} \rightarrow 0} \frac{2}{u^3} \left[-\frac{u}{1+\Lambda^2 u^2} + \frac{\pi/2}{\Lambda} \right] = 0$. Moreover using $Lt_{\Lambda^{-1} \rightarrow 0} \int_0^{\infty} \Omega \cos(\Omega uv) e^{-\Omega/\Lambda} = \frac{\Lambda^{-1} - u^2 v^2}{(\Lambda^{-2} + u^2 v^2)^2} = \frac{-1}{u^2 v^2}$, the expression for the current becomes

$$J = -\frac{e^2 E}{\hbar \pi^2} \int_0^{\infty} du \int_0^1 dv \frac{1}{u^2 v^2} \frac{u}{2} (1+v^2) e^{-\delta u} \left[e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)} - e^{-\frac{\bar{E}^2 u^3}{12}} \right] \quad (104)$$

$$+ \frac{e^2 E}{\hbar \pi^2} \left(\frac{2\bar{T}_{eff}}{\pi} \right) \int_0^{\infty} du \int_0^1 dv \frac{1}{uv} \sin(\bar{T}_{eff} uv) \frac{u}{2} (1+v^2) e^{-\delta u} e^{-\frac{\bar{E}^2 u^3}{12} (1+6v^2-3v^4)}$$

As before we discuss the following two cases

A. Quantum Disordered Regime, $\delta^R \gg \bar{E}$. In this regime we find

$$J = \frac{e^2 E}{\hbar \pi^2} \left[\frac{8}{15} \frac{\bar{E}^2}{\delta^3} + \frac{4}{3\pi} \frac{\bar{T}_{eff}^2}{\delta^2} \right] \quad (105)$$

The second term above is the correction to the results of⁹ due to the effective temperature of the nonequilibrium electrons.

B. Quantum Critical Regime, $\delta^R \ll \bar{E}$. Here we obtain the result

$$J = \frac{e^2 E}{\hbar \pi^2} \left[\frac{1}{6} \int_0^1 dv \frac{(1+v^2)}{v^2} \ln(1+6v^2-3v^4) + \frac{\bar{T}_{eff}^2}{\pi \bar{E}^{4/3}} \int_0^{\infty} du \int_0^1 dv u (1+v^2) e^{-\frac{u^3}{12} (1+6v^2-3v^4)} \right] \quad (106)$$

Computing the above integrals we find

$$J = \frac{0.46 e^2 E}{\hbar} \left[1 + 0.82 \frac{\bar{T}_{eff}^2}{\bar{E}^{4/3}} \right] \quad (107)$$

The first term is the universal conductivity found in⁹, while the second term is the contribution due to current noise. As discussed after Eq. 98, this correction is of $\mathcal{O}((T_{eff} \tau_{sc})^{2/3})$ and is therefore subleading for this model of electric field independent τ_{sc} .

It is instructive to see how the current due to superconducting fluctuations in the quantum critical regime get modified for a $1d$ system. The steps in the derivation are the same except that there is only one momentum integral in Eq. 100. We find

$$J_{1d} = \frac{e^2 E \sqrt{\gamma}}{\hbar \pi \sqrt{\pi}} \left[\bar{E}^{1/3} \frac{2^{2/3}}{3^{1/6}} \Gamma\left(\frac{5}{6}\right) \int_0^1 dv \frac{1+v^2}{v^2} (g^{1/6}(v) - 1) + \frac{2\bar{T}_{eff}^2}{\pi \bar{E}} \int_0^{\infty} du \int_0^1 dv \sqrt{u} (1+v^2) e^{-\frac{u^3}{12} g(v)} \right] \quad (108)$$

where $g(v) = 1 + 6v^2 - 3v^4$.

Eq. 108 shows that unlike 2d, the response to the electric field in the quantum critical regime is highly nonlinear, with $J_{1d} \propto E^{4/3}$. Current noise here too gives subleading corrections of $\mathcal{O}((T_{eff} \tau_{sc})^{2/3})$.

The results presented above are for the case of $\delta > 0$, i.e, the system is on the normal side in equilibrium. The case of $\delta < 0$ and large electric fields so that one is on the current/supercurrent induced disordered side can be analyzed by employing a purely classical Ginzburg-Landau theory corresponding to a temperature $T = T_{eff}$. The computation of the non-linear response would follow²⁵, where using their results one expects the fluctuation current in dimension d to be, $J_d \propto \frac{T_{eff}}{\bar{E}^{(4-d)/3}} E$.

VII. CONCLUSIONS

In summary starting from a fermionic model under external drive, we have presented a microscopic derivation of the effect of current flow on a superconducting order-parameter. Our microscopic treatment reveals that current besides directly coupling to the order-parameter also produces a noise and a drift of the order-parameter, the origin of which is the underlying nonequilibrium electron gas. We study the effects of these new terms on scaling near the equilibrium quantum critical point. Scaling equations when only the direct coupling between the order-parameter and the electric field is present was derived by Dalidovich and Phillips⁹ in a phenomenological approach. Here we find that current drift gives a small correction of $\mathcal{O}(1/E_F\tau_{sc})$ to their result. Current noise on the other hand gives corrections that are of $\mathcal{O}((T_{eff}\tau_{sc})^{2/3})$ in the quantum critical regime. In our model where τ_{sc} is independent of the electric field, this correction is subdominant to the effect of the direct coupling between the order-parameter and the electric-field. In the quantum disordered regime however the noise and direct-coupling effects are found to be equally dominant.

One may easily imagine a scenario where noise effects dominate over direct-coupling effects both in the quantum-critical and quantum disordered regime. This would occur when $\tau_{sc} \sim T_{eff}^{-p}$ where $p > 1$, a physical situation for this being when the dominant inelastic scattering mechanism is due to phonons. There are several experiments involving electric-field scaling in thin films near a superconducting transition^{12,26,27}. As discussed in¹², the results of many of these experiments can be explained only when taking into account noise effects due to a nonequilibrium electron gas. For example $p = 2$ for electron-phonon coupled MoGe thin films, clearly making $T_{eff}\tau_{sc} \gg 1$ in these systems.

Our derivation is valid on the normal side and outside the Ginzburg regime. Extension of the results of this paper to the nonequilibrium ordered side is currently in progress.

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APPENDIX A: EFFECTIVE EQUILIBRIUM ACTION FOR FLUCTUATIONS ABOUT THE ORDERED STATE

In order to understand the fluctuational properties on the ordered side in the absence of an applied electric field, the action will be derived for a partition function,

$$Z = \int \mathcal{D}[\Delta, \Delta^*] \exp \left(- \int d\tau d^d r \frac{|\Delta|^2}{\lambda} + Tr \ln \mathcal{G}^{-1} \right) \quad (A1)$$

where in terms of a complex $\Delta = \Delta_0 e^{2i\theta}$, $\Delta^* = \Delta_0 e^{-2i\theta}$

$$\mathcal{G}^{-1} = \begin{pmatrix} -\partial_\tau - \frac{1}{2m} \left(\frac{\vec{\nabla}}{i} - \frac{e}{c} \vec{A} \right)^2 - \Sigma + \mu & \Delta_0 e^{2i\theta} \\ \Delta_0 e^{-2i\theta} & -\partial_\tau + \frac{1}{2m} \left(\frac{\vec{\nabla}}{i} + \frac{e}{c} \vec{A} \right)^2 - \bar{\Sigma} - \mu \end{pmatrix} \quad (A2)$$

$\Sigma = \bar{\Sigma}$ are the self-energies due to coupling to the underlying metallic substrate with

$$\Sigma(\tau) = \frac{\Gamma}{\pi} P \left(\frac{1}{\tau} \right) \quad (A3)$$

The action may be written as an expansion in fluctuations in the magnitude Δ_0 and phase θ of the order parameter. In what follows we will consider only fluctuations in the phase as the fluctuations in the magnitude of Δ are gapped in the ordered phase. To this end it is convenient to introduce the unitary matrix $U = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, and transform the Green's function as

$$\mathcal{G}^{-1} \rightarrow U \mathcal{G}^{-1} U^\dagger = \quad (A4)$$

$$\begin{pmatrix} -\partial_\tau - i\phi - \frac{1}{2m} \left(\frac{\vec{\nabla}}{i} - \frac{e}{c} \vec{A} \right)^2 - e^{-i\theta} \Sigma e^{i\theta} + \mu & \Delta_0 \\ \Delta_0 & -\partial_\tau + i\phi + \frac{1}{2m} \left(\frac{\vec{\nabla}}{i} + \frac{e}{c} \vec{A} \right)^2 - e^{i\theta} \bar{\Sigma} e^{-i\theta} - \mu \end{pmatrix} \quad (A5)$$

$$= \mathcal{G}_0^{-1} + X_{1a} + X_{1b} + X_2 \quad (A6)$$

where $\phi = \partial_\tau \theta$, $\vec{A} = \vec{A} - \frac{e}{c} \vec{\nabla} \theta$ and we have split the above terms as follows

$$\mathcal{G}_0^{-1} = \begin{pmatrix} i\omega_n - \xi_k + i\Gamma \text{sgn}(\omega_n) & \Delta_0 \\ \Delta_0 & i\omega_n + \xi_k + i\Gamma \text{sgn}(\omega_n) \end{pmatrix} \quad (\text{A7})$$

$$X_{1a} = -i\sigma_3 \phi + \frac{i}{2m} \sigma_0 \left\{ \vec{\nabla}, \frac{e}{c} \vec{A} \right\}_+ \quad (\text{A8})$$

$$X_{1b} = \Sigma - e^{-i\theta\sigma_3} \Sigma e^{i\theta\sigma_3} \quad (\text{A9})$$

$$X_2 = -\sigma_3 \frac{e^2}{2mc^2} \vec{A}^2 \quad (\text{A10})$$

Expanding to quadratic order in the fluctuations, the action for the superconductor takes the form

$$Z = \int \mathcal{D}\theta \exp \left(- \int d\tau d^d r \left[c_1 (\partial_\tau \theta)^2 + c_2 \left(\vec{\nabla} \theta - \frac{e}{c} \vec{A} \right)^2 \right] + S_\Sigma \right) \quad (\text{A11})$$

where the first two terms above are the usual ones that arise in any superconductor with the coefficients changed due to coupling to an underlying substrate. In particular,

$$c_1 = -\frac{1}{2} \frac{1}{\beta L^d} \text{Tr} [\mathcal{G}_0 \sigma_3 \mathcal{G}_0 \sigma_3] \quad (\text{A12})$$

$$c_2 = \frac{n_s}{2m} - \frac{1}{2m^2 d} \frac{1}{\beta L^d} \text{Tr} [p^2 \mathcal{G}_0 \sigma_0 \mathcal{G}_0 \sigma_0], \quad d = \text{dimension} \quad (\text{A13})$$

The new feature is S_Σ which arises specifically due to coupling to external normal metal reservoirs and reflects the lack of gauge invariance associated with the non-conservation of particle number in the superconducting layer. To leading order in the fluctuation of the phase,

$$S_\Sigma = \text{Tr} [\mathcal{G} X_{1b}] = \text{Tr} \left[(G(xt, xt') \Sigma(t', t) + \Sigma(t, t') \bar{G}(xt', xt)) \left(1 - e^{-i\theta(x, t')} e^{i\theta(x, t)} \right) \right] \quad (\text{A14})$$

Evaluating the above trace we obtain a Caldeira-Leggett type local damping,

$$S_\Sigma = g \int d^d x \frac{1}{\beta} \sum_m |\Omega|_m (e^{-i\theta(x)})_{\Omega_m} \left(e^{i\theta(x)} \right)_{-\Omega_m} \quad (\text{A15})$$

where

$$g = 4\Gamma \left(\frac{\nu\Gamma}{\sqrt{\Delta_0^2 + \Gamma^2}} \right) \quad (\text{A16})$$

Fourier transforming Eq. A15 one gets,

$$S_\Sigma = \frac{g}{2\pi} \int d^d x \int d\tau \int d\tau' \left(\frac{\theta(x, \tau) - \theta(x, \tau')}{\tau - \tau'} \right)^2 \quad (\text{A17})$$

APPENDIX B: DERIVATION OF STEADY STATE SINGLE PARTICLE GREEN'S FUNCTIONS

1. Derivation of retarded Green's functions

To obtain the retarded Green's function in the presence of an electric field and coupling to an external reservoir we need to solve the Dyson equation,

$$[i\partial_{t_1} - H_0(t_1)] G_0^R(t_1 - t_2) = \delta(t_1 - t_2) + \Sigma^R G_0^R \quad (\text{B1})$$

where $H_0(t) = \sum_{\vec{k}_\perp \sigma} \epsilon \left(\vec{k}_\perp - \frac{e\vec{A}(t)}{\hbar c} \right) \psi_{\vec{k}_\perp \sigma}^\dagger \psi_{\vec{k}_\perp \sigma}$, $\vec{A} = -c\vec{E}t$ and

$$\Sigma^R(k_\perp, \omega) = \sum_{k_z} \frac{t_{k_z}^2}{\omega - \epsilon_{k_z, k_\perp}^b + i\delta} \quad (\text{B2})$$

For energy independent tunneling amplitude, density of states, and using the fact that $\epsilon_{k_z, k_\perp}^b = \epsilon_{k_z}^b + \epsilon_{k_\perp}^b$ the above expression simplifies to give an energy independent self-energy

$$\Sigma^R(\omega) = -i\pi t^2 \rho \int d\epsilon_z^b \delta(\omega - \epsilon_z^b - \epsilon_{k_\perp}^b) = -i\Gamma \quad (\text{B3})$$

The above implies

$$\Sigma^R(t_1, t_2) = -i\Gamma \delta(t_1 - t_2) \quad (\text{B4})$$

Substituting the above in Eq. B1, it is straightforward to show that the retarded Green's function in the presence of an electric field and coupling to leads is:

$$G_0^R(\vec{k}, \tau) = -i\theta(\tau) e^{-\frac{i}{\hbar} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} dx \epsilon(\vec{k} + \frac{e}{\hbar} \vec{E}x)} e^{-\Gamma\tau} \quad (\text{B5})$$

where $\tau = t_1 - t_2$ and $k = p + eET$, (where we set $\hbar = 1$). The above time integral in the argument may be performed to obtain the following series expansion

$$G_0^R(\vec{k}, \tau) = -i\theta(\tau) e^{-i\epsilon_k \tau - \frac{i\tau^3}{24} (e\vec{E} \cdot \frac{\partial}{\partial \vec{k}})^2 \epsilon_k + \dots} e^{-\Gamma\tau} \quad (\text{B6})$$

Now we define

$$T_{eff} = eE v_F \tau_{sc} \quad (\text{B7})$$

$$\tau_{sc}^{-1} = 2\Gamma \quad (\text{B8})$$

and $E_F = v_F/a$ with a being the lattice spacing. Then, the second term in the argument of the exponent in Eq. B6 is $(eE)^2 (\tau_{sc}^3 \partial^2 \epsilon_k / (\partial k^2)) = \frac{\partial^2 \epsilon / a^2 \partial k^2}{E_F} \frac{(T_{eff} \tau_{sc})^2}{E_F \tau_{sc}} \ll 1$ and therefore may be neglected. A similar argument applies to the higher order terms.

Thus, we may approximate the retarded Green's function by its value in the absence of an electric field,

$$G_0^R(\vec{k}, \tau) = -i\theta(\tau) e^{-i\epsilon_k \tau} e^{-\Gamma\tau} \quad (\text{B9})$$

provided k is chosen to be the canonical momentum.

2. Derivation of steady state Keldysh Green's function

The equation of motion obeyed by the Keldysh Green's function is

$$\left(i \vec{\partial}_{t_1} - H_0 \right) G_0^K(t_1, t_2) = 1 + \Sigma^R G_0^K + \Sigma^K G_0^A \quad (\text{B10})$$

$$G_0^K(t_1, t_2) \left(-i \overleftarrow{\partial}_{t_2} - H_0 \right) = 1 + G_0^R \Sigma^K + G_0^K \Sigma^A \quad (\text{B11})$$

Taking the difference between the equations B10 and B11, one obtains

$$(i\partial_{t_1} + i\partial_{t_2}) G_0^K(t_1, t_2) - \epsilon(t_1) G_0^K(t_1, t_2) + \epsilon(t_2) G_0^K(t_1, t_2) = \Sigma^R G_0^K + \Sigma^K G_0^A - G_0^R \Sigma^K - G_0^K \Sigma^A \quad (\text{B12})$$

The solution for G_0^K may be obtained by using the ansatz

$$G_0^K = G_0^R f_K - f_K G_0^A \quad (\text{B13})$$

where $1 - 2f = f_K$, with f the generalized distribution function. The equation of motion for f^K is

$$i \frac{\partial f_K}{\partial t_1} + i \frac{\partial f_K}{\partial t_2} - \epsilon_{p - \frac{e}{\hbar c} A(t_1)} f_K + \epsilon_{p - \frac{e}{\hbar c} A(t_2)} f^K - \Sigma^R \cdot f_K + f_K \cdot \Sigma^A + \Sigma^K = 0$$

$\Sigma^R - \Sigma^A = -\frac{i}{\tau_{sc}}$ and $\Sigma^K = (\Sigma^R - \Sigma^A)(1 - 2g)$, g being the distribution function of the substrate. Fourier transforming Eq. B14 with respect to the relative time $\tau = t_1 - t_2$, changing variables to the canonical momentum $\vec{k} = \vec{p} + e\vec{E}T$ and expanding in E one finds that the distribution function at steady state obeys,

$$e\vec{E} \cdot \frac{\partial f}{\partial \vec{k}} + \frac{\partial f}{\partial \omega} \left(e\vec{E} \cdot \frac{\partial \epsilon_k}{\partial \vec{k}} \right) + \frac{1}{24} \frac{\partial^3 f}{\partial \omega^3} \left(e\vec{E} \cdot \frac{\partial}{\partial \vec{k}} \right)^3 \epsilon_k \dots = \frac{1}{\tau_{sc}} [-f + g] \quad (\text{B14})$$

The usual quasiclassical arguments imply that the first term in Eq B14 is negligible while in the the weak field limit the third term may be dropped. With these simplifications we find

$$f = f^s + f^a \quad (\text{B15})$$

where

$$f_{k,x}^s = \theta(-x) + \frac{\text{sign}(x)}{2} e^{-\frac{|x|}{\sqrt{(e\vec{E} \cdot \vec{v}_k \tau_{sc})^2}}} \quad (\text{B16})$$

$$f_{k,x}^a = \frac{(e\vec{E} \cdot \vec{v}_k \tau_{sc})}{2\sqrt{(e\vec{E} \cdot \vec{v}_k \tau_{sc})^2}} e^{-\frac{|x|}{\sqrt{(e\vec{E} \cdot \vec{v}_k \tau_{sc})^2}}} \quad (\text{B17})$$

where $x = \omega - \mu$ and $v_k = \partial \varepsilon_k / \partial k$. Substitution of Eqs B16, B17 into Eq B14 then shows that the neglect of the third term in Eq B14 is justified when the coupling of the layer to the substrate is sufficiently weak ($\frac{\partial^3 \epsilon_k}{E_F a^3 \partial k^3} \ll (E_F \tau_{sc})^2$) while the first term is negligible in the weak field limit $T_{eff} \ll E_F^2 / (\frac{\partial^2 \epsilon_k}{a^2 \partial k^2})$.

APPENDIX C: EVALUATION OF THE POLARIZATION BUBBLES WHEN $E \neq 0$

The retarded and Keldysh polarization bubbles may be expressed as an expansion in $(\vec{q} + 2e\vec{E}T)^2$ and Ω as shown in Eq. 62, 63. In particular,

$$\tilde{\Pi}^R(\vec{q} + 2\vec{E}T = 0, \Omega) = \quad (\text{C1})$$

$$i \sum_{\vec{k}} \int \frac{d\omega}{2\pi} \left[G_{0R}(\vec{k}, \omega + \Omega) G_{0K}(-\vec{k}, -\omega) + G_{0K}(\vec{k}, \omega + \Omega) G_{0R}(-\vec{k}, -\omega) \right]$$

$$\Pi^K(\vec{q} + 2\vec{E}T = 0, \Omega) = \quad (\text{C2})$$

$$i \sum_{\vec{k}} \int \frac{d\omega}{2\pi} \left[G_{0K}(\vec{k}, \omega + \Omega) G_{0K}(-\vec{k}, -\omega) + G_{0R}(\vec{k}, \omega + \Omega) G_{0R}(-\vec{k}, -\omega) + G_{0A}(\vec{k}, \omega + \Omega) G_{0A}(-\vec{k}, -\omega) \right]$$

Using Eq. B9, B13, Eq. B15, Eq. B16 B17 we find

$$Im \left[\tilde{\Pi}^R(\Omega) \right] = \frac{i\Omega}{2} \sum_{\vec{p}} \int \frac{d\omega}{2\pi} \frac{(-2i\Gamma)^2}{((\omega - \xi_p)^2 + \Gamma^2)((\omega + \xi_p)^2 + \Gamma^2)} \frac{1}{\sqrt{(e\vec{E} \cdot \vec{v}_F \tau_{sc})^2}} e^{-\frac{|\omega|}{\sqrt{(e\vec{E} \cdot \vec{v}_F \tau_{sc})^2}}} \quad (\text{C3})$$

where $2\Gamma = \tau_{sc}^{-1}$. For $eEv_F \tau_{sc} < 1/\tau_{sc}$, the above expression simplifies to

$$Im \left[\Pi^R(\Omega) \right] = -2i\Omega\nu\tau_{sc} \quad (\text{C4})$$

In the same way, one finds

$$\Pi^K(\Omega) = -4i\nu\tau_{sc} \left[|\Omega| + T_{eff} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} |\cos \phi| e^{-\frac{|\Omega|}{T_{eff} |\cos \phi|}} \right] \quad (\text{C5})$$

It is also instructive to derive the expressions for the polarization bubbles in 1d. While $\Pi^{R,A}$ have the same structure as in 2d, the noise Π^K has the form,

$$\Pi_{1d}^K(\Omega) = -4i\nu\tau_{sc} \left[|\Omega| + T_{eff} e^{-\frac{|\Omega|}{T_{eff}}} \right] \quad (\text{C6})$$

Note that the above expressions were derived and used to study the effect of current flow on magnetic fluctuations in⁸.

In order to evaluate the coefficient of expansion in powers of $q + 2eET$ in Eq. 65 we first write the expression for the polarization bubble in momentum-time space

$$\Pi^R(\vec{q} + 2\vec{E}T, t_1 - t_2) = i \sum_{\vec{k}} \left[G_{0R}(\vec{k} + \vec{q} + 2e\vec{E}T, t_1 - t_2) G_{0K}(-\vec{k}, t_1 - t_2) + G_{0K}(\vec{k} + \vec{q} + 2e\vec{E}T, t_1 - t_2) G_{0R}(-\vec{k}, t_1 - t_2) \right] \quad (C7)$$

For convenience, we shift variables so that $q + 2eET$ appears in the argument of the retarded functions so that

$$\Pi^R(\vec{q} + 2\vec{E}T, t_1 - t_2) = i \sum_{\vec{k}} \left[G_{0R}(\vec{k} + \vec{q} + 2e\vec{E}T, t_1 - t_2) G_{0K}(-\vec{k}, t_1 - t_2) + G_{0K}(\vec{k}, t_1 - t_2) G_{0R}(-\vec{k} + \vec{q} + 2e\vec{E}T, t_1 - t_2) \right] \quad (C8)$$

Using Eq. B5, the retarded Green's function can be expanded in a power series in $Q = q + 2eET$ as follows,

$$G_{0R}(\vec{k} + \vec{Q}, \tau) = G_{0R}(\vec{k}, \tau) \left[1 - i \frac{\vec{k} \cdot \vec{Q}}{m} \tau - i \frac{Q^2}{2m} \tau - \frac{1}{2} \left(\frac{\vec{Q} \cdot \vec{k}}{m} \right)^2 \tau^2 + \dots \right] \quad (C9)$$

In the above we assume quadratic dispersion. Note that in equilibrium, the linear in Q term does not survive the angle integration. On the other hand, a non-zero current picks a preferred direction so that this term for our case will no longer be zero. As we shall show, this term will give rise to current-drift.

Fourier transforming Eq. C9 with respect to τ we get

$$G_{0R}(k + Q, \omega) = \left[1 - \frac{\vec{k} \cdot \vec{Q}}{m} \frac{\partial}{\partial \omega} - \frac{Q^2}{2m} \frac{\partial}{\partial \omega} + \frac{1}{2} \left(\frac{\vec{Q} \cdot \vec{k}}{m} \right)^2 \frac{\partial^2}{\partial \omega^2} \right] G_{0R}(k, \omega) \quad (C10)$$

Thus Eq. C10 and C8 lead to the following for the particle-hole symmetric case,

$$\lambda (\Pi^R(Q, 0) - \Pi^R(0, 0)) = \gamma Q^2 - \frac{i\lambda\nu}{2\Gamma} \left(\frac{e\vec{E} \cdot \vec{Q}}{m} \right) \tau_{sc} \quad (C11)$$

where

$$\gamma = -\frac{2\lambda\nu}{\pi} \frac{\mu}{m} \int d\xi \int d\omega \text{sgn}(\omega) \left(\frac{\Gamma}{(\omega + \xi_k)^2 + \Gamma^2} \right) \left[\frac{(\omega - \xi_k)^3 - 3\Gamma^2(\omega - \xi_k)}{((\omega - \xi_k)^2 + \Gamma^2)^3} \right] = \frac{\lambda\nu\mu}{4m\Gamma^2} \quad (C12)$$

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